Lecture 7. Linear systems via linear transformations

Prop Every linear system can be written as $A\vec{x} = \vec{b}$ where

- A is the matrix with coefficients as entries,
- $\vec{b}$ is the vector with constant terms as entries.

e.g. $\begin{cases} x_1 \qquad\quad + 3x_3 = 2 \\ 2x_1 - 3x_2 + \ \ x_3 = 0 \\ \qquad\quad x_2 - 2x_3 = 4 \end{cases} \Longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -3 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}}_{\vec{b}}$

Note The matrix that represents the linear system is not equal to A, but is given by attaching $\vec{b}$ to A as the last column.

e.g. $\begin{cases} x_1 \qquad\quad + 3x_3 = 2 \\ 2x_1 - 3x_2 + \ \ x_3 = 0 \\ \qquad\quad x_2 - 2x_3 = 4 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 2 & -3 & 1 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad A \qquad\quad \vec{b}$

Def Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.

(1) T is injective (or one-to-one) if we have $T(\vec{u}) \neq T(\vec{v})$ for any distinct $\vec{u}, \vec{v} \in \mathbb{R}^n$.

(2) T is surjective (or onto) if every vector in $\mathbb{R}^m$ is the $\underset{\text{output}}{\underline{\text{image}}}$ of some vector in $\mathbb{R}^n$.

e.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$\Longrightarrow \begin{cases} T \text{ is not injective: } T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T \text{ is not surjective: } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$

<u>Thm</u> Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A.

(1) T is injective $\iff$ RREF(A) has a leading 1 in every column

(2) T is surjective $\iff$ RREF(A) has a leading 1 in every row

<u>pf</u> (1) T is injective

$\iff T(\vec{x}) \neq T(\vec{0}) = \vec{0}$ for any $\vec{x} \neq \vec{0}$

$\iff A\vec{x} \neq \vec{0}$ for any $\vec{x} \neq \vec{0}$

$\iff A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} \text{RREF}(A) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$$

$\iff$ RREF(A) has a leading 1 in every column (no free variables)

(2) T is surjective

$\iff T(\vec{x}) = \vec{b}$ has solutions for any $\vec{b} \in \mathbb{R}^m$

$\iff A\vec{x} = \vec{b}$ has solutions for any $\vec{b} \in \mathbb{R}^m$

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} \text{RREF}(A) & \begin{matrix} * \\ \vdots \\ * \end{matrix} \end{array}\right]$$

$\iff$ RREF(A) has no zero rows

(to avoid a leading 1 in the last column)

$\iff$ RREF(A) has a leading 1 in every row

Ex Let $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 4 & 1 & 0 & -1 \\ 1 & 0 & 2 & -2 \\ 2 & 1 & -4 & 3 \end{bmatrix}$$

(1) Parametrize all vectors whose image under T is $\vec{0}$.

Sol We solve the equation $T(\vec{x}) = \vec{0} \Rightarrow A\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 4 & 1 & 0 & -1 & 0 \\ 1 & 0 & 2 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & -8 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A $\vec{0}$

$$\begin{cases} x_1 + 2x_3 - 2x_4 = 0 \\ x_2 - 8x_3 + 7x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_3 + 2x_4 \\ x_2 = 8x_3 - 7x_4 \end{cases}$$

Set $x_3 = s$ and $x_4 = t$ (free variables)

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + 2t \\ 8s - 7t \\ s \\ t \end{bmatrix} = \boxed{s \begin{bmatrix} -2 \\ 8 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -7 \\ 0 \\ 1 \end{bmatrix}}$$

(2) Determine whether the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

is the image of some vector under T.

Sol We consider the equation $T(\vec{x}) = \vec{v} \Rightarrow A\vec{x} = \vec{v}$

$$\left[\begin{array}{cccc|c} 4 & 1 & 0 & -1 & 1 \\ 1 & 0 & 2 & -2 & 0 \\ 2 & 1 & -4 & 3 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & -8 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

a leading 1 in the last column

A $\vec{v}$

The equation has a solution.

$\Rightarrow \vec{v}$ is $\boxed{\text{not the image of any vector under } T}$

(3) Determine whether T is injective.

Sol From (1) and (2), we find

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -8 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\Rightarrow$ RREF(A) has no leading 1s in column 3 and column 4

$\Rightarrow$ T is $\boxed{\text{not injective}}$

Note We can get the same answer from (1) as the equation $T(\vec{x}) = \vec{0}$ has infinitely many solutions.

(4) Determine whether T is surjective.

Sol RREF(A) has no leading 1s in row 3

$\Rightarrow$ T is $\boxed{\text{not surjective}}$

Note We can get the same answer from (2) as $\vec{v}$ is not the image of any vector under T.